Quaternionic vector coherent states for spin-orbit interactions

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Abstract

This work addresses the study of two-dimensional electron gas in a perpendicular magnetic field in the presence of both Rashba and Dresselhaus spin-orbit (SO) interactions, with effective Zeeman coupling. Exact analytical expressions are found for the eigenvalue problem giving the Landau levels and the associated eigenstates, thanks to an appropriate method of parametrization used to quantize the physical system. Such a system exhibits interesting properties of quaternions for which vector coherent states are built. Similarity with the Weyl-Heisenberg group and some relevant properties of these vectors are discussed.

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1 Introduction

Spin orbit Hamiltonian systems continue to attract increasing interest in recent years due to their prevalence in various physical applications as in semiconductors and spintronics. For details in relevant applications, see [1] and references therein. To cite here a few, let us mention that in the paradigmatic Datta-Das spin transistor, the spin of the electron passing through the device is controlled by the Rashba spin-orbit (SO) interaction [2], which in turn can be varied by the application of gate voltages. The Rashba interaction stems from the structural inversion asymmetry (SIA) introduced by a heterojunction or by surface or external fields. In semiconductors with narrower energy gap (InGaAs, AlGaAs), this effect is expected to be stronger. It has been shown experimentally that the Rashba spin-orbit interaction can be modified up to 50% by external gate voltages [3], [4]. In addition to the Rashba coupling, there is also a material-intrinsic Dresselhaus spin-orbit interaction. This originates from the bulk inversion asymmetry (BIA) of the crystal, and can be relatively large in semiconductors like InSb/InAlSb [5]. Both Rashba and Dresselhaus interactions contribute to the spin-dependent splitting of the band structure of the host material, leading to dramatic spin-dependent phenomena for electrons or holes in semiconductors.

Globally, the determination of the eigenvalues and eigenstates of such systems is crucial for the understanding and the computation of a number of their important physical properties [6]. All these aspects motivate the investigations of the properties of the SO Hamiltonian systems. Recently, standard and nonlinear vector coherent states (VCS) have been constructed for relevant generalized models in quantum optics for nonlinear spin-Hall effect [7] and for the Jaynes-Cummings model in the rotating wave approximation [8]. See also references quoted in these works. Despite all these investigations, it is worthy of attention to note that only a few exactly analytically solvable eigenvalue problems associated to SO interactions exist in the literature. Besides, in the most interesting case of the description of the more general case, in which both Rashba and Dresselhaus interactions are present with arbitrary strength, in the effective Zeeman coupling, exact solutions to the eigenvalue problem are still lacking to our best knowledge of the existing works. The present paper aims, among others, at fulfilling such a gap. We give explicit exact analytical solutions to the addressed eigenvalue problem and build quaternionic VCS (QVCS) for the spinor Landau levels. Similarity properties in connection with the Weyl-Heisenberg group is examined.

The paper is organized as follows. Section 2 deals with the model description in two gauge symmetries corresponding to the two orientations of the magnetic field. Then, the eigenvalue problem is addressed and solved. In section 3, VCS and QVCS are formulated and some physical properties of the latter states are investigated. Then, follows the conclusion.

2 Physical Hamiltonians

2.1 Hamiltonian with Zeeman coupling

The Hamiltonian of a particle system with mass M and charge e with Zeeman coupling g, in an electromagnetic field can be defined by

$$\mathcal{H} = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r},\mathbf{t})\right)^2 - g\frac{\mu}{2}\sigma \cdot \mathbf{B} \tag{1}$$

where in the term $g\mu/2\sigma.\mathbf{B}$, the symbol $\mu = e\hbar/2M$ denotes the Bohr magneton, g the factor of coupling related to the particle and $\sigma = (\sigma_x, \sigma_y, \sigma_z) \equiv (\sigma_1, \sigma_2, \sigma_3)$ the vector of Pauli spin matrices.

Let us first consider the gauge symmetry $\mathbf{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right)$. Then, the generalized Hamiltonian (1) becomes for an electron with charge $-e$,

$$\mathcal{H}_{1_0}(\mathbf{p}, \mathbf{r}) \equiv \mathcal{H}_{1_0} = \frac{1}{2M}\left[\left(p_x - \frac{eB}{2c}y\right)^2 + \left(p_y + \frac{eB}{2c}x\right)^2\right] - \frac{g\mu B}{2}\sigma_z. \tag{2}$$

Following the general scheme of canonical quantization, we can introduce the coordinate and momentum operators R_i, P_i, $i = 1, 2$ corresponding to the classical coordinates x, y and their conjugate momenta p_x, p_y, respectively, so that the canonical commutation relations

$$[R_i, P_j] = \imath\hbar\delta_{ij}, \quad \text{for} \quad R_i = X, Y \quad \text{and} \quad P_i = P_x, P_y, \ i = 1, 2 \tag{3}$$

hold. Using the change of variables

$$Z = X + \imath Y, \qquad P_z = \frac{1}{2}(P_x - \imath P_y), \tag{4}$$

and the creation and annihilation operators $b^\dagger, b$ defined as

$$b^\dagger = -2\imath P_{\bar{z}} + \frac{eB}{2c}Z, \qquad b = 2\imath P_z + \frac{eB}{2c}\bar{Z}, \tag{5}$$

and such that

$$[b, b^\dagger] \ = \ 2M\hbar\omega_c \tag{6}$$

where

$\omega_c = \frac{eB}{Mc}$ (the cyclotron frequency), the Hamiltonian (2) can be reexpressed in the manner

$$H_{1_0} = \hbar\omega_c\left(b'^\dagger b' + \frac{\mathbb{I}}{2}\right)\sigma_0 + \hbar\omega_c\xi\sigma_z, \quad \xi = -g\mu Mc/2\hbar e, \tag{7}$$

with $\sigma_0 = \mathbb{I}_2$ the identity matrix, where the annihilation and creation operators b and $b^\dagger$ have been simplified as follows:

$$b = \sqrt{2M\hbar\omega_c}b', \qquad b^\dagger = \sqrt{2M\hbar\omega_c}b'^\dagger, \tag{8}$$

with

$$[b', b'^\dagger] = \mathbb{I}. \tag{9}$$

The eigenvalues related to H_{1_0} are given by

$$\mathcal{E}_n^\pm = \hbar\omega_c\left(n + \frac{1}{2}\right) \mp \hbar\omega_c\xi. \tag{10}$$

The related eigenstates, denoted by $\begin{pmatrix} |\Phi_n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |\Phi_n\rangle \end{pmatrix}$, span the Hilbert space $\hat{\mathfrak{H}} := \mathbb{C}^2 \otimes \mathfrak{H}$, with $\mathfrak{H} := span\{|\Phi_n\rangle\}_{n=0}^\infty$, $|\Phi_n\rangle$ corresponding to the number operator $N = b'^\dagger b'$ orthonormalized eigenstates $|n\rangle = (1/\sqrt{n!})(b'^\dagger)^n|0\rangle$, and

$$\hat{\mathfrak{H}} = \left\{|\chi^+ \otimes \Phi_n\rangle = \begin{pmatrix} |\Phi_n\rangle \\ 0 \end{pmatrix}, \ |\chi^- \otimes \Phi_n\rangle = \begin{pmatrix} 0 \\ |\Phi_n\rangle \end{pmatrix}, \ n = 0, 1, 2 \ldots\right\}, \tag{11}$$

where χ^+ and χ^-, given by $\chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, form an orthonormal basis of $\mathbb{C}^2$.

In opposite, in the gauge symmetry $\mathbf{A} = \left(\frac{B}{2}y, -\frac{B}{2}x\right)$, the Hamiltonian is given by

$$\mathcal{H}_{2_0}(\mathbf{p},\mathbf{r}) \equiv \mathcal{H}_{2_0} = \frac{1}{2M}\left[\left(p_x + \frac{eB}{2c}y\right)^2 + \left(p_y - \frac{eB}{2c}x\right)^2\right] + \frac{g\mu B}{2}\sigma_z, \tag{12}$$

with the same change of variables as in (4). The creation and annihilation operators given, respectively, by

$$\hat{d}^\dagger = 2\imath P_z - \frac{eB}{2c}\bar{Z}, \qquad \hat{d} = -2\imath P_{\bar{z}} - \frac{eB}{2c}Z \tag{13}$$

satisfy the following commutation relations

$$[\hat{d}, \hat{d}^\dagger] \;=\; 2M\hbar\omega_c. \tag{14}$$

The corresponding canonically quantized Hamiltonian can be written as

$$H_{2_0} = \hbar\omega_c\left(\hat{d}^\dagger\hat{d} + \frac{\mathbb{I}}{2}\right)\sigma_0 + \hbar\omega_c\xi'\sigma_z, \tag{15}$$

where $\xi' = -\xi$, through the annihilation and creation operators $\hat{d}$ and $\hat{d}^\dagger$ built as follows:

$$\hat{d} = \sqrt{2M\hbar\omega_c}\hat{d}', \qquad \hat{d}^\dagger = \sqrt{2M\hbar\omega_c}\hat{d}'^\dagger \tag{16}$$

with

$$[\hat{d}', \hat{d}'^\dagger] = \mathbb{I}. \tag{17}$$

The eigenvalues and eigenstates related to the Hamiltonian H_{2_0} are obtained as previously.

2.2 Eigenvalues and eigenstates of general spin-orbit interactions Hamiltonians

Through a reparametrization using the Fock operators or by making use of projections and defining the Pauli matrices $\sigma_\pm = (\sigma_x \pm \imath\sigma_y)/2$, the generalization of Rashba H_R and Dresselhaus H_D SO interactions can be introduced as:

$$H_R = \imath\hbar\omega_c\gamma\left(\lambda(N)(b'^\dagger)^k\sigma_- - b'^k\overline{\lambda(N)}\sigma_+\right), H_D = \hbar\omega_c\beta\left(b'^k\lambda(N)\sigma_- + \overline{\lambda(N)}(b'^\dagger)^k\sigma_+\right) \tag{18}$$

such that, for two values of the parameter ε, there follow

(i) $\varepsilon = +$,

$$B^+_{k,+} = b'^{-k}\lambda(N) = b'^k\lambda(N), \quad B^-_{k,+} = \overline{\lambda(N)}b'^{+k} = \overline{\lambda(N)}(b'^\dagger)^k; \tag{19}$$

(ii) $\varepsilon = -$,

$$B^{+}_{k,-} = b'^{+k}\lambda(N) = \left(b'^{\dagger}\right)^{k}\lambda(N), \quad B^{-}_{k,-} = -\overline{\lambda(N)}b'^{-k} = -\overline{\lambda(N)}b'^{k}, \tag{20}$$

where $\lambda(N)$ is a nonvanishing complex valued function of the number operator N, $\overline{\lambda(N)}$ denotes its complex conjugate and k is a strictly positive integer, $k = 1, 2, \ldots$. The expressions in (i) and (ii) can be abridged as

$$B^{+}_{k,\varepsilon} = b'^{-\varepsilon k}\lambda(N), \qquad B^{-}_{k,\varepsilon} = \varepsilon\overline{\lambda(N)}b'^{\varepsilon k}, \tag{21}$$

and denoting the generalized dimensionless SO interaction by

$$V_{k,\varepsilon} = B^{+}_{k,\varepsilon}\sigma_{-} + B^{-}_{k,\varepsilon}\sigma_{+}, \tag{22}$$

the relations (18) take the following compact forms

$$H_R = \imath\hbar\omega_c\gamma V_{k,\varepsilon}, \qquad\qquad H_D = \hbar\omega_c\beta V_{k,\varepsilon}. \tag{23}$$

Recall that in the study of a spin-1/2 particle of charge $-e$ subjected to a spin-orbit interaction in a perpendicular magnetic field $\mathbf{B} = -B\hat{z} = \nabla \times \mathbf{A}$ and an electric field $\mathbf{E} = E\hat{y}$ along the y axis [6], using the introduced lowering operator a, the both Rashba and Dresselhaus couplings are given by $V_R = \imath\sqrt{2}\eta_R(a\sigma_{-} - a^{\dagger}\sigma_{+})$ and $V_D = \sqrt{2}\eta_D(a^{\dagger}\sigma_{-} + a\sigma_{+})$ which lead to the generalized dimensionless spin-orbit interaction established in [7] as: $U_{k,\epsilon} = C^{+}_{k,\epsilon}\sigma_{+} + C^{-}_{k,\epsilon}\sigma_{-}$, where $C^{+}_{k,\epsilon} = a^{-\epsilon k}\lambda^{\epsilon}(N)$, $C^{-}_{k,\epsilon} = \lambda^{-\epsilon}(N)a^{\epsilon k}$, with $\epsilon = \pm 1$.

From (22) and (23), the generalized Hamiltonians describing a $2D$ electron gas mixing with full Rashba and Dresselhaus SO interactions, respectively, and effective Zeeman coupling in a perpendicular magnetic field background is given in both relevant gauge symmetries by

$$H_{1,2_{SO}} = \frac{1}{2M}\left[\left(p_x \pm \frac{eB}{2c}\right)^2 + \left(p_y \mp \frac{eB}{2c}\right)^2\right] \mp \frac{g\mu B}{2}\sigma_z + V_{SO}, \tag{24}$$

where V_{SO} stands for the generalized Rashba or Dresselhaus spin-orbit terms given in (23).

Remark 2.1 *The relevant exactly solvable cases known in the literature [9] correspond to $\beta = 0$, $\gamma = 0$ and $\gamma = \pm\beta$ and concern with the gauge in which the constant magnetic field points along the positive $z-$ direction. The eigenstates and eigenvalues of the associated resulting Hamiltonians are given by the expressions:*

- *for $\beta = 0$ and nonvanishing Rashba SO interaction term, which couples the up-spin level $(\Phi_{n-1}, 0)$ to the down-spin state $(0, \Phi_n), n \geq 1$*

$$\psi^{+}_{n} = \begin{pmatrix} \cos\theta_n\Phi_{n-1} \\ \sin\theta_n\Phi_n \end{pmatrix}, \qquad \psi^{-}_{n} = \begin{pmatrix} -\sin\theta_n\Phi_{n-1} \\ \cos\theta_n\Phi_n \end{pmatrix}$$
$$E^{\pm}_{n} = \hbar\omega_c n \mp \frac{\delta}{2}\sqrt{1 + 4\gamma^2 n\hbar^2\omega_c^2/\delta^2}, \tag{25}$$

in which $\delta = \hbar\omega_c(1 + 2\xi)$. The mixing angle θ_n is given by $\tan 2\theta_n = 2\sqrt{n}\gamma\hbar\omega_c/\delta$ and varies from 0 (when $\gamma = 0$) to $\pi/4$ (for weak magnetic field or strong SO interaction). Each Landau level $\Phi_n \uparrow\downarrow$ splits into two levels ψ^{-}_{n} and ψ^{+}_{n+1} with the splitting gap $\Delta = E^{-}_{n} - E^{+}_{n+1}$,

- *for $\gamma = 0$ and nonvanishing Dresselhaus SO interaction term which couples $(\Phi_n, 0)$ only to the $(0, \Phi_{n-1}), n \geq 1$*

$$\psi_n^+ = \begin{pmatrix} \cos\theta_n \Phi_n \\ \sin\theta_n \Phi_{n-1} \end{pmatrix}, \qquad \psi_n^- = \begin{pmatrix} -\sin\theta_n \Phi_n \\ \cos\theta_n \Phi_{n-1} \end{pmatrix}$$
$$E_n^{\pm} = \hbar\omega_c n \mp \frac{\delta}{2}\sqrt{1 + 4\beta^2 n\hbar^2\omega_c^2/\delta^2}, \tag{26}$$

with $\delta = \hbar\omega_c(-1+2\xi)$, $\tan 2\theta_n = 2\sqrt{n}\beta\hbar\omega_c/\delta$.

It is also worth noticing that for the most usual widespread situation corresponding to $\lambda(N) = -\imath\mathbf{1}$, $k = 1$, the cases of $\gamma = \pm\beta$ and $|\beta| < |\gamma|$ have been treated with or without the Zeeman coupling term, using the perturbation theory. For more details, see [1]. In the latter case, the Rashba and Dresselhaus SO interactions take the simpler forms in the gauge $\mathbf{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right)$, where H_R and H_D are obtained from (19) and (20) by

$$H_R = \imath\hbar\omega_c\gamma(-\imath b'^{\dagger}\mathbf{1}\sigma_- - \imath b'\mathbf{1}\sigma_+) = \hbar\omega_c\gamma(b'^{\dagger}\sigma_- + b'\sigma_+) = \hbar\omega_c\gamma\begin{pmatrix} 0 & b' \\ b'^{\dagger} & 0 \end{pmatrix}, \tag{27}$$

$$H_D = \hbar\omega_c\beta(\imath b'^{\dagger}\mathbf{1}\sigma_+ - \imath b'\mathbf{1}\sigma_-) = \hbar\omega_c\beta(\imath b'^{\dagger}\sigma_+ - \imath b'\sigma_-) = \hbar\omega_c\beta\begin{pmatrix} 0 & \imath b'^{\dagger} \\ -\imath b' & 0 \end{pmatrix}, \tag{28}$$

respectively. γ and β are dimensionless parameters defined by $\gamma = \gamma_0\sqrt{\frac{2M}{\hbar^3\omega_c}}$ and $\beta = \beta_0\sqrt{\frac{2M}{\hbar^3\omega_c}}$.

In the sequel, we turn back to the more general situation when the interaction terms (27) and (28) are coupled with the generalized Hamiltonians H_{1_0} and H_{2_0} investigated in the previous section, describing a 2D electron in electromagnetic field with a Zeeman coupling term.

Analog expressions are readily obtained for $H_{2_{SO}} = H_{2_0} + V_{SO}$ in the second gauge, when the constant magnetic field points along the negative $z-$ direction, by replacing δ by δ' defined with ξ'.

The spinor states are formulated as

- for $\beta = 0$ and $\gamma \neq 0$

$$\begin{aligned}
\psi_n^+ &= \cos\theta_n \begin{pmatrix} \Phi_{n-1} \\ 0 \end{pmatrix} \oplus \sin\theta_n \begin{pmatrix} 0 \\ \Phi_n \end{pmatrix} \\
&= \cos\theta_n(\chi^+ \otimes \Phi_{n-1}) \oplus \sin\theta_n(\chi^- \otimes \Phi_n),
\end{aligned}$$

$$\begin{aligned}
\psi_n^- &= -\sin\theta_n \begin{pmatrix} \Phi_{n-1} \\ 0 \end{pmatrix} \oplus \cos\theta_n \begin{pmatrix} 0 \\ \Phi_n \end{pmatrix} \\
&= -\sin\theta_n(\chi^+ \otimes \Phi_{n-1}) \oplus \cos\theta_n(\chi^- \otimes \Phi_n),
\end{aligned} \tag{29}$$

- for $\beta \neq 0$ and $\gamma = 0$

$$\psi_n^+ = \cos\theta_n \begin{pmatrix} \Phi_n \\ 0 \end{pmatrix} \oplus \sin\theta_n \begin{pmatrix} 0 \\ \Phi_{n-1} \end{pmatrix}$$

$$= \cos\theta_n(\chi^+ \otimes \Phi_n) \oplus \sin\theta_n(\chi^- \otimes \Phi_{n-1}),$$

$$\begin{aligned}\psi_n^- &= -\sin\theta_n \begin{pmatrix} \Phi_n \\ 0 \end{pmatrix} \oplus \cos\theta_n \begin{pmatrix} 0 \\ \Phi_{n-1} \end{pmatrix} \\ &= -\sin\theta_n(\chi^+ \otimes \Phi_n) \oplus \cos\theta_n(\chi^- \otimes \Phi_{n-1}). \end{aligned} \tag{30}$$

In the weak Rashba SO coupling limit case (γ negligible), we can expand $\mathcal{E}_n^\pm$ using the approximation $(1+\varepsilon)^{1/2} \simeq 1 + \varepsilon/2, \varepsilon \ll 1$ and, by taking $\hbar = 1$, get from (25)

$$\mathcal{E}_n^\pm = \omega_\pm(\gamma)n, \qquad \omega_\pm(\gamma) = \omega_c \left(1 \mp \frac{2\gamma^2}{2 - gc}\right) \tag{31}$$

where $gc \ll 2$ is assumed. Define

$$\rho_\pm(n) := \mathcal{E}_1^\pm \mathcal{E}_2^\pm \cdots \mathcal{E}_n^\pm \tag{32}$$

with the increasing order $\mathcal{E}_1^\pm < \mathcal{E}_2^\pm < \cdots < \mathcal{E}_n^\pm$, such that

$$\rho_\pm(n) = \prod_{q=1}^{n} \omega_\pm(\gamma) q = n!(\omega_\pm(\gamma))^n. \tag{33}$$

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2.3 Spectral decomposition

Introduce the passage operators from the Hilbert space $\mathcal{S}$ with eigenbasis $\{|\psi_n^\pm\rangle\}$ to $\{|\chi^\pm \otimes \Phi_n\rangle\}$ and vice versa, $\chi^\pm$ being the natural basis of $\mathbb{C}^2$, given by

$$\mathcal{U}|\chi^\pm \otimes \Phi_n\rangle = |\psi_n^\pm\rangle \qquad \mathcal{U}^\dagger|\psi_n^\pm\rangle = |\chi^\pm \otimes \Phi_n\rangle \tag{34}$$

where $\mathcal{U}, \mathcal{U}^\dagger$ expand as

$$\mathcal{U} = \sum_{n=0,\pm}^{\infty} |\psi_n^\pm\rangle\langle\chi^\pm \otimes \Phi_n| \qquad \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle\langle\psi_n^\pm| \tag{35}$$

respectively. $\mathcal{U}, \mathcal{U}^\dagger$ are obtained as mutually adjoint through the following identities:

$$\mathcal{U}^\dagger\mathcal{U} = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle\langle\chi^\pm \otimes \Phi_n| = \mathbb{I}_2 \otimes I_{\mathfrak{H}}, \qquad \mathcal{U}\mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |\psi_n^\pm\rangle\langle\psi_n^\pm| = \mathbb{I}_{\mathcal{S}}, \tag{36}$$

where $\mathbb{I}_2 \otimes I_{\mathfrak{H}}$ and $\mathbb{I}_{\mathcal{S}}$ are the identity on $\mathbb{C}^2 \otimes \mathfrak{H}$ and $\mathcal{S}$, respectively. Then, the Hamiltonians $H_{1,2_{SO}}$ can be rewritten in a diagonal form as below:

$$\mathbb{H}_{1,2_{SO}}^{red} = \mathcal{U}^\dagger H_{1,2_{SO}} \mathcal{U} = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle \mathcal{E}_n^\pm \langle\chi^\pm \otimes \Phi_n|. \tag{37}$$

3 Canonical QVCS and VCS

In this section, we first provide the canonical QVCS related to the Hamitonian $\mathbb{H}_{1_{SO}}^{red}$ and then discuss the VCS construction with their connection with the QVCS. Then, statistical properties of the constructed QVCS are investigated through studying some expectations and the uncertainty relation.

3.1 Canonical QVCS

The QVCS are built using the complex representation of quaternions by 2×2 matrices [10]. Using the basis matrices,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \imath\sigma_1 = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}, \; -\imath\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; \imath\sigma_3 = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix}, \tag{38}$$

where σ_1, σ_2 and σ_3 are the usual Pauli matrices, a general quaternion is written as

$$q = x_0\sigma_0 + \imath\underline{x}.\underline{\sigma} \tag{39}$$

with $x_0 \in \mathbb{R}$, $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\underline{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$. Thus,

$$\mathfrak{q} = \begin{pmatrix} x_0 + \imath x_3 & -x_2 + \imath x_1 \\ x_2 + \imath x_1 & x_0 - \imath x_3 \end{pmatrix}. \tag{40}$$

Introducing the polar coordinates as:
$x_0 = r\cos\eta,\; x_1 = r\sin\eta\sin\phi\cos\psi,\; x_2 = r\sin\eta\sin\phi\sin\psi,\; x_3 = r\sin\eta\cos\phi$, where $r \in [0,\infty), \phi \in [0,\pi]$ and $\eta, \psi \in [0, 2\pi]$, it follows that

$$\mathfrak{q} = A(r)e^{\imath\eta\sigma(\hat{n})} \tag{41}$$

where

$$A(r) = r\sigma_0, \qquad \sigma(\hat{n}) = \begin{pmatrix} \cos\phi & e^{\imath\psi}\sin\phi \\ e^{-\imath\psi}\sin\phi & -\cos\phi \end{pmatrix} \qquad \text{and} \qquad \sigma(\hat{n})^2 = \sigma_0 = \mathbb{I}_2. \tag{42}$$

The field of quaternions is denoted by $\mathbb{H}$, this algebra being generated by $\{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$, where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are imaginary units. The real quaternions denoted by $Q_{\mathbb{R}}$ are given by

$$Q_{\mathbb{R}} = \{q = x_0 + x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k} |\; \hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = -1,\; \hat{\imath}\hat{\jmath}\hat{k} = -1,\; x_0, x_1, x_2, x_3 \in \mathbb{R}\}, \tag{43}$$

where q can be written as $q = q(x_0, \underline{x}) = x_0 + \underline{i}.\underline{x}$, with $\underline{i} = (\hat{\imath}, \hat{\jmath}, \hat{k})$ and $\underline{x} = (x_1, x_2, x_3)$ given as in (39). The complex quaternions denoted by $Q_{\mathbb{C}}$ are given by $q = x_0 + \imath x_1\hat{\imath} + \imath x_2\hat{\jmath} + \imath x_3\hat{k} = x_0 + \imath\underline{i}.\underline{x}$.

Remark that in the definition of the Hamiltonian given in (7), the harmonic oscillator part of H_{1_0}, given by $\hbar\omega_c\left(b'^{\dagger}b' + \frac{\mathbf{1}}{2}\right)\sigma_0$, can be rewritten in the supersymmetric quantum mechanics as

$$H^{SUSY} = \begin{pmatrix} H^b & 0 \\ 0 & H^f \end{pmatrix} = \hbar\omega_c \begin{pmatrix} b'^{\dagger}b' & 0 \\ 0 & b'b'^{\dagger} \end{pmatrix}, \tag{44}$$

where the supersymmetric partner Hamiltonians, H^b, H^f are given by $H^b = H_{1_{OSC}} - \frac{\hbar\omega_c}{2}\mathbb{I}$ and $H^f = H_{1_{OSC}} + \frac{\hbar\omega_c}{2}\mathbb{I}$, with $H_{1_{OSC}} = \hbar\omega_c\left(b'^\dagger b' + \frac{\mathbb{I}}{2}\right)$, respectively. This Hamiltonian can also be rewritten as $H^{SUSY} = \{Q, Q^\dagger\}$, with $Q = \sqrt{\hbar\omega_c}\begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} = \sqrt{\hbar\omega_c}\, b'\sigma_-$ and $Q^\dagger$ the supercharges. The annihilation operator defined on $\mathbb{C}^2 \otimes \mathfrak{H}$ by

$$\mathcal{A}' = \begin{pmatrix} b' & 0 \\ 0 & b' \end{pmatrix} \tag{45}$$

satisfies the commutation relation $[H^{SUSY}, \mathcal{A}'] = -\hbar\omega_c\mathcal{A}'$. The connection with the supersymmetric harmonic oscillator in this study appears by obtaining the QVCS denoted by $|\mathfrak{q};\pm\rangle$, known in the literature as the *quaternionic canonical coherent states* [10], given on $\mathbb{C}^2 \otimes \mathfrak{H}$ as follows

$$|\mathfrak{q};\pm\rangle \;=\; \frac{e^{-\frac{r^2}{2}}}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{\mathfrak{q}^n}{\sqrt{n!}}|\chi^\pm \otimes \Phi_n\rangle, \tag{46}$$

where $\mathfrak{q}$ is given in (41), as the eigenstates of the operator $\mathcal{A}'$ in (45).

The QVCS related to the Hamiltonian $\mathbb{H}^{red}_{1_{SO}}$ can be defined as

$$|\mathfrak{q};\pm\rangle_1 \;=\; \left(\mathcal{N}_{(\star)}(|\mathfrak{q}|)\right)^{-1/2}\sum_{n=0}^{\infty}\frac{\mathfrak{q}^n}{\sqrt{\rho_{(\star)}(n)}}|\chi^\pm \otimes \Phi_n\rangle, \tag{47}$$

where $\mathcal{N}_{(\star)}(|\mathfrak{q}|), \rho_{(\star)}(n)$ stand for $\mathcal{N}_+(|\mathfrak{q}|)$ and $\rho_+(n)$ (resp. $\mathcal{N}_-(|\mathfrak{q}|)$ and $\rho_-(n)$). The variables $\mathfrak{q}$ are as in (41). The 2×2 matrix-valued functions $A(r), \sigma(\hat{n})$ satisfy the following properties [10] (assumed to hold almost everywhere with respect to appropriate measures):

$$\sigma(\hat{n}) = \sigma(\hat{n})^\dagger \qquad [A(r), A(r)^\dagger] = 0 \qquad [A(r), \sigma(\hat{n})] = 0. \tag{48}$$

The normalization constants $\mathcal{N}_{(\star)}(|\mathfrak{q}|)$ ensured by the relations

$$\sum_{\pm} {}_1\langle\mathfrak{q};\pm|\mathfrak{q};\pm\rangle_1 = 1, \tag{49}$$

are given, with respect to (48) for $|\mathfrak{q};\pm\rangle_1$ by

$$\mathcal{N}_{(\star)}(|\mathfrak{q}|) = \mathcal{N}_{(\star)}(r) \;=\; 2\sum_{n=0}^{\infty}\frac{r^{2n}}{n!(\omega_{(\star)}(\gamma))^n} = 2e^{\frac{r^2}{\omega_{(\star)}(\gamma)}}. \tag{50}$$

The $\mathbb{H}^{red}_{1_{SO}}$ QVCS (47) satisfy on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$ the following resolution of the identity

$$\sum_{\pm}\int_{\mathcal{D}}|\mathfrak{q};\pm\rangle_1\, W_{(\star)}(r)\, {}_1\langle\mathfrak{q};\pm|\, d\nu \;=\; \mathbb{I}_2 \otimes I_{\mathfrak{H}}, \tag{51}$$

$W_{(\star)}(r)$ is the density function and $\mathcal{D} = \mathbb{R}^+ \times [0, 2\pi) \times S^2$, where S^2 is the surface of the unit two-sphere on which the points with coordinates (ϕ, ψ) are defined. The measure $d\nu$ is such that $d\nu(r, \eta, \phi, \psi) = r dr d\eta d\Omega(\phi, \psi)$, with $d\Omega(\phi, \psi) = \frac{1}{4\pi}\sin\phi d\phi d\psi$.

The relation (51) translates in the Stieljes moment problems

$$\int_0^{\infty} r^{2n+1}\varrho_{(\star)}(r)dr = \rho_{(\star)}(n), \tag{52}$$

which are solved by

$$\varrho_{(\star)}(r) = \frac{2}{\omega_{(\star)}(\gamma)} e^{-\frac{r^2}{\omega_{(\star)}(\gamma)}} \tag{53}$$

where the density functions are given by

$$W_{(\star)}(r) = \frac{\mathcal{N}_{(\star)}(r)\varrho_{(\star)}(r)}{2\pi}. \tag{54}$$

Treating the vectors $|\mathfrak{q};+\rangle_1$ and $|\mathfrak{q};-\rangle_1$ as elements of a basis, we shall define a general QVCS for $\mathbb{H}^{red}_{1_{SO}}$ (resp. $\mathbb{H}^{red}_{2_{SO}}$) as a linear combination,

$$|\mathfrak{q},\chi\rangle = c_+|\mathfrak{q};+\rangle_1 + c_-|\mathfrak{q};-\rangle_1 \tag{55}$$

where $c_\pm \in \mathbb{C}, |c_+|^2 + |c_-|^2 = 1$ and $\chi = \sum_\pm c_\pm \chi^\pm$.

It is worth noticing that the constructed QVCS are related to the Weyl-Heisenberg group G_{W-H} as observed in [10]. Indeed, from the relations

$$e^{\mathfrak{q}\otimes b'^\dagger - \mathfrak{q}^\dagger \otimes b'} = e^{(-1/2)[\mathfrak{q}\otimes b'^\dagger, -\mathfrak{q}^\dagger\otimes b']} e^{\mathfrak{q}\otimes b'^\dagger} e^{-\mathfrak{q}^\dagger\otimes b'}, \quad [\mathfrak{q}^\dagger\otimes b', \mathfrak{q}\otimes b'^\dagger] = r^2 \mathbb{I}_2 \otimes I_{\mathfrak{H}} \tag{56}$$

since

$$e^{-\mathfrak{q}\otimes b'}|\chi^\pm \otimes \Phi_0\rangle = |\chi^\pm\otimes\Phi_0\rangle, \tag{57}$$

we can write

$$\frac{1}{\sqrt{2}} e^{\frac{1}{\omega_{(\star)}(\cdot)}[\mathfrak{q}\otimes b'^\dagger - \mathfrak{q}^\dagger\otimes b']}|\chi^\pm\otimes\Phi_0\rangle \;=\; \frac{e^{-\frac{r^2}{2\omega_{(\star)}(\cdot)}}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\mathfrak{q}^n}{\sqrt{n!(\omega_{(\star)}(\cdot))^n}}|\chi^\pm\otimes\Phi_n\rangle, \tag{58}$$

where $\omega_{(\star)}(\cdot)$ stands for $\omega_{(\star)}(\beta)$ and $\omega_{(\star)}(\gamma)$, respectively. Therefore, in the case of the Hamiltonian $H_{1_{SO}}$, the QVCS can be rewritten as

$$|\mathfrak{q};\pm\rangle_1 = \frac{1}{\sqrt{2}} U_1^{(\star)}(0,\mathfrak{q})|\chi^\pm\otimes\Phi_0\rangle \qquad |\mathfrak{q};\pm\rangle_1^{\sim} = \frac{1}{\sqrt{2}} \tilde{U}_1^{(\star)}(0,\mathfrak{q})|\chi^\pm\otimes\Phi_0\rangle, \tag{59}$$

in analogy with the case of the canonical coherent states [11], where the operators b' and $b'^\dagger$ are identical to those given in (8).

The unitary operator $\tilde{U}$ has the form [10]

$$\tilde{U}(0,\mathfrak{q}) = e^{\mathfrak{q}\otimes b'^\dagger - \mathfrak{q}^\dagger\otimes b'} = u(\eta,\phi)\begin{pmatrix} U(0,q,p) & 0 \\ 0 & U(0,q,-p)\end{pmatrix} u(\eta,\phi)^\dagger, \tag{60}$$

and for fixed $(\eta,\phi), \eta,\phi, \in [0,\pi]$ and p,q, such that $z = \frac{q-\imath p}{\sqrt{2}}$, the operators $\tilde{U}_1^{(\star)}(0,\mathfrak{q})$ and $\tilde{U}_2^{(\star)}(0,\mathfrak{q})$ are defined for Rashba and Dresselhaus SO interactions, respectively, by

$$U_1^{(\star)}(0,\mathfrak{q}) = e^{\frac{1}{\omega_{(\star)}(\gamma)}[\mathfrak{q}\otimes b'^\dagger - \mathfrak{q}^\dagger\otimes b']}, \qquad U_2^{(\star)}(0,\mathfrak{q}) = e^{\frac{1}{\omega_{(\star)}(\gamma)}[\mathfrak{q}\otimes \hat{d}'^\dagger - \mathfrak{q}^\dagger\otimes \hat{d}']}, \tag{61}$$

$$\tilde{U}_1^{(\star)}(0,\mathfrak{q}) = e^{\frac{1}{\omega_{(\star)}(\beta)}[\mathfrak{q}\otimes b'^\dagger - \mathfrak{q}^\dagger\otimes b']}, \qquad \tilde{U}_2^{(\star)}(0,\mathfrak{q}) = e^{\frac{1}{\omega_{(\star)}(\beta)}[\mathfrak{q}\otimes \hat{d}'^\dagger - \mathfrak{q}^\dagger\otimes \hat{d}']}. \tag{62}$$

They realize unitary (reducible) representations of the Weyl-Heisenberg group G_{W-H} [10] on the Hilbert space $\mathbb{C}^2\otimes\mathfrak{H}$.

3.2 Vector coherent states related to $\mathbb{H}^{red}_{1_{SO}}$

This paragraph deals with the Hamiltonian $\mathbb{H}^{red}_{1_{SO}}$ VCS and QVCS construction. Each set of vectors satisfies a resolution of the identity on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$.

The VCS are given on $\mathbb{C}^2 \otimes \mathfrak{H}$ as

$$|Z; \pm\rangle_1 = (\mathcal{N}(Z))^{-1/2} \sum_{n=0}^{\infty} (R(n))^{-1/2} Z^n |\chi^{\pm} \otimes \Phi_n\rangle \tag{63}$$

where $Z = \mathrm{diag}(z_1, z_2), z_1 = r_1 e^{i\varphi_1}, z_2 = r_2 e^{i\varphi_2}, r_1, r_2 \in [0, \infty), \varphi_1, \varphi_2 \in [0, 2\pi)$ and $R(n) = \mathrm{diag}(\rho_+(n), \rho_-(n))$.

In this case, the normalization factor is given by

$$\mathcal{N}(Z) = e^{r_1^2/\omega_+(\gamma)} + e^{r_2^2/\omega_-(\gamma)}. \tag{64}$$

They fulfill on $\mathbb{C}^2 \otimes \mathfrak{H}$ the following resolution of the identity

$$\sum_{\pm} \int_{\mathcal{D}_1 \times \mathcal{D}_2} |Z; \pm\rangle_1 W(r_1, r_2)\, {}_1\langle Z; \pm| d\mu = \mathbb{I}_2 \otimes I_{\mathfrak{H}} \tag{65}$$

where the measure $d\mu$ is given on $\mathcal{D}_1 \times \mathcal{D}_2 = \{[0, \infty) \times [0, 2\pi)\}^2$ by $d\mu(r_1, r_2, \varphi_1, \varphi_2) = \frac{1}{4\pi^2} r_1 dr_1 r_2 dr_2 d\varphi_1 d\varphi_2$. From (53) and (54), the density function is now provided by

$$W(r_1, r_2) = \mathcal{N}(Z) \frac{4 e^{-r_1^2/\omega_+(\gamma)} e^{-r_2^2/\omega_-(\gamma)}}{\omega_+(\gamma)\omega_-(\gamma)}. \tag{66}$$

Let $\mathcal{Z} = diag(z, \bar{z})$ where $z = \tilde{r} e^{\imath\vartheta}, \tilde{r} \in [0, \infty), \vartheta \in [0, 2\pi)$. Consider $\mathfrak{Z} = U \mathcal{Z} U^{\dagger}, U \in SU(2)$ with $U = u_{\phi_1} u_{\varphi} u_{\phi_2}$, $\phi_1, \phi_2 \in [0, 2\pi)$ and $\varphi \in [0, \pi]$, where

$$u_{\phi_j} = \begin{pmatrix} e^{\imath \frac{\phi_j}{2}} & 0 \\ 0 & e^{-\imath \frac{\phi_j}{2}} \end{pmatrix}, \; j = 1, 2, \quad u_{\varphi} = \begin{pmatrix} \cos\frac{\varphi}{2} & \imath \sin\frac{\varphi}{2} \\ \imath \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}. \tag{67}$$

Introduce the quaternion $\mathfrak{Q} = B(\tilde{r}) e^{\imath\vartheta\tilde{\sigma}(\hat{k})}$ with $B(\tilde{r}) = \tilde{r}\mathbb{I}_2$ and

$$\tilde{\sigma}(\hat{k}) = \begin{pmatrix} \cos\varphi & e^{\imath\varrho} \sin\varphi \\ e^{-\imath\varrho} \sin\varphi & -\cos\varphi \end{pmatrix} \tag{68}$$

and $\varphi \in [0, \pi], \vartheta, \varrho \in [0, 2\pi]$. For $\phi_1 = \phi_2 = \varrho$ we get $\mathfrak{Z} = \tilde{r}(\mathbb{I}_2 \cos\vartheta + \imath\tilde{\sigma}(\hat{k}) \sin\vartheta) = \mathfrak{Q}$. Then, the QVCS are given by

$$|\mathfrak{Q}; \pm\rangle_1 = (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} (R(n))^{-1/2} \mathfrak{Q}^n |\chi^{\pm} \otimes \Phi_n\rangle. \tag{69}$$

The normalization factor is provided by

$$\mathcal{N}(\tilde{r}) = e^{\tilde{r}^2/\omega_+(\gamma)} + e^{\tilde{r}^2/\omega_-(\gamma)}. \tag{70}$$

They satisfy on $\mathbb{C}^2 \otimes \mathfrak{H}$ the following resolution of the identity

$$\sum_{\pm} \int_{\mathcal{D}} |\mathfrak{Q}; \pm\rangle_1 W(\tilde{r})\, {}_1\langle \mathfrak{Q}; \pm| d\mu = \mathbb{I}_2 \otimes I_{\mathfrak{H}} \tag{71}$$

where $\mathcal{D} = [0, \infty) \times [0, 2\pi) \times S^2$ and $d\mu = \frac{1}{4\pi} \tilde{r} d\tilde{r} \sin\varphi d\varphi d\varrho d\vartheta$ with

$$W(\tilde{r}) = \begin{pmatrix} \frac{\mathcal{N}(\tilde{r})}{\pi} \frac{(2\omega_c)^{n+1}}{(\omega_-(\gamma))^n} \frac{e^{-\tilde{r}^2\left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)}\right]}}{\omega_+(\gamma)\omega_-(\gamma)} & 0 \\ 0 & \frac{\mathcal{N}(\tilde{r})}{\pi} \frac{(2\omega_c)^{n+1}}{(\omega_+(\gamma))^n} \frac{e^{-\tilde{r}^2\left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)}\right]}}{\omega_+(\gamma)\omega_-(\gamma)} \end{pmatrix}. \tag{72}$$

Proof.

$$\begin{aligned}
&\sum_{\pm} \int_{\mathcal{D}} |\mathfrak{Q}; \pm\rangle_1 W(\tilde{r})\, {}_1\langle \mathfrak{Q}; \pm| d\mu \\
&= \int_{\mathcal{D}} W(\tilde{r}) \times \\
&\sum_{\pm} \left| (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} \frac{\mathfrak{Q}^n \chi^{\pm} \otimes \Phi_n}{\sqrt{R(n)}} \right\rangle \left\langle (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{m=0}^{\infty} \frac{\mathfrak{Q}^m \chi^{\pm} \otimes \Phi_m}{\sqrt{R(m)}} \right| d\mu \\
&= \sum_{\pm} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{\mathcal{D}} \frac{W(\tilde{r})}{\mathcal{N}(\tilde{r})\sqrt{R(m)R(n)}} \mathfrak{Q}^n |\chi^{\pm}\rangle\langle \chi^{\pm}| \mathfrak{Q}^{\dagger m} \otimes |\Phi_n\rangle\langle \Phi_m| d\mu \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{\mathcal{D}} \frac{W(\tilde{r})}{\mathcal{N}(\tilde{r})\sqrt{R(m)R(n)}} B(\tilde{r})^n e^{in\vartheta\tilde{\sigma}(\hat{k})} \left(\sum_{\pm} |\chi^{\pm}\rangle\langle \chi^{\pm}| \right) \\
&\times B(\tilde{r})^{m\dagger} e^{-im\vartheta\tilde{\sigma}(\hat{k})\dagger} \otimes |\Phi_n\rangle\langle \Phi_m| d\mu.
\end{aligned}$$

Using the facts

$$\begin{aligned}
&\sum_{\pm} |\chi^{\pm}\rangle\langle \chi^{\pm}| = \mathbb{I}_2, \\
&\tilde{\sigma}(\hat{k})^{\dagger} = \tilde{\sigma}(\hat{k}) \qquad \text{and} \\
&\int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi} e^{i(n-m)\vartheta\tilde{\sigma}(\hat{k})} \sin\varphi d\varphi d\varrho d\vartheta = \begin{cases} 0 & \text{if} \quad m \neq n, \\ 8\pi^2 \mathbb{I}_2 & \text{if} \quad m = n, \end{cases}
\end{aligned} \tag{73}$$

the last line is written as

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{\mathcal{D}} \frac{W(\tilde{r})}{\mathcal{N}(\tilde{r})\sqrt{R(m)R(n)}} B(\tilde{r})^n e^{in\vartheta\tilde{\sigma}(\hat{k})} \left(\sum_{\pm} |\chi^{\pm}\rangle\langle \chi^{\pm}| \right) \\
&\times B(\tilde{r})^{m\dagger} e^{-im\vartheta\tilde{\sigma}(\hat{k})\dagger} \otimes |\Phi_n\rangle\langle \Phi_m| d\mu \\
&= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{2\pi W(\tilde{r})}{\mathcal{N}(\tilde{r})} |B(\tilde{r})|^{2n} \mathrm{diag}\left(\frac{1}{\rho_+(n)}, \frac{1}{\rho_-(n)} \right) \otimes |\Phi_n\rangle\langle \Phi_m| \tilde{r} d\tilde{r}.
\end{aligned} \tag{74}$$

Taking

$$W(\tilde{r}) = \begin{pmatrix} \frac{\mathcal{N}(\tilde{r})}{\pi} \frac{(2\omega_c)^{n+1}}{(\omega_-(\gamma))^n} \frac{e^{-\tilde{r}^2\left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)}\right]}}{\omega_+(\gamma)\omega_-(\gamma)} & 0 \\ 0 & \frac{\mathcal{N}(\tilde{r})}{\pi} \frac{(2\omega_c)^{n+1}}{(\omega_+(\gamma))^n} \frac{e^{-\tilde{r}^2\left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)}\right]}}{\omega_+(\gamma)\omega_-(\gamma)} \end{pmatrix} \tag{75}$$

we get from (74) that

$$\begin{aligned}
&\sum_{\pm}\int_{\mathcal{D}}|\mathfrak{Q};\pm\rangle_1 W(\tilde r)\,{}_1\langle\mathfrak{Q};\pm|d\mu \\
=&\sum_{n=0}^{\infty}\begin{pmatrix} \frac{(2\omega_c)^{n+1}}{(\omega_-(\gamma))^n}\int_0^\infty \frac{2\tilde r^{2n}}{\rho_+(n)}\frac{e^{-\tilde r^2\left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)}\right]}}{\omega_+(\gamma)\omega_-(\gamma)}\tilde r d\tilde r & 0 \\ 0 & \frac{(2\omega_c)^{n+1}}{(\omega_+(\gamma))^n}\int_0^\infty \frac{2\tilde r^{2n}}{\rho_-(n)}\frac{e^{-\tilde r^2\left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)}\right]}}{\omega_+(\gamma)\omega_-(\gamma)}\tilde r d\tilde r \end{pmatrix} \\
&\times\mathbb{I}_2\otimes|\Phi_n\rangle\langle\Phi_n| \\
=&\ \mathbb{I}_2\otimes\sum_{n=0}^{\infty}|\Phi_n\rangle\langle\Phi_n| = \mathbb{I}_2\otimes I_{\mathfrak{H}}
\end{aligned}$$

where the following moment problems

$$\frac{(2\omega_c)^{n+1}}{(\omega_\mp(\gamma))^n}\int_0^\infty \frac{2\tilde r^{2n}}{\rho_\pm(n)}\frac{e^{-\tilde r^2\left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)}\right]}}{\omega_+(\gamma)\omega_-(\gamma)}\tilde r d\tilde r = 1 \tag{76}$$

are satisfied.

□

3.3 Physical properties

In this section, we study the physical properties of the QVCS (47) and (69). The generalized annihilation, creation and number operators are introduced and allow the definitions of the quadrature operators. Their actions on the QVCS (47) and (69) are then derived. Next, the expectations of these operators are given in the second paragraph. The last paragraph discusses the time evolution of the constructed QVCS.

3.3.1 Generalized annihilation, creation and number operators

The generalized annihilation, creation and number operators, given on the Hilbert space $\mathfrak{H}$, with respect to the basis $\{|\Phi_n\rangle\}_{n=0}^{\infty}$ can be defined as:

$$a|\Phi_n\rangle = \sqrt{x_n}|\Phi_{n-1}\rangle, \quad a^\dagger|\Phi_n\rangle = \sqrt{x_{n+1}}|\Phi_{n+1}\rangle, \quad N'|\Phi_n\rangle = x_n|\Phi_n\rangle, \tag{77}$$

with $x_n = n\omega_{(\star)}(\cdot)$ such that $x_n! = n!(\omega_{(\star)}(\cdot))^n, x_0! = 1$.

The corresponding operators for the QVCS (47) are given on $\mathbb{C}^2\otimes\mathfrak{H}$ by

$$A = \mathbb{I}_2\otimes a, \quad A^\dagger = \mathbb{I}_2\otimes a^\dagger, \quad N = \mathbb{I}_2\otimes N'. \tag{78}$$

Their actions on the the QVCS (47) are

$$A|\mathfrak{q};\pm\rangle_1 = \mathfrak{q}|\mathfrak{q};\pm\rangle_1, \tag{79}$$

$$A^\dagger|\mathfrak{q};\pm\rangle_1 = \left(\mathcal{N}_{(\star)}(|\mathfrak{q}|)\right)^{-1/2}\sum_{n=0}^{\infty}\mathfrak{q}^n\sqrt{\frac{x_{n+1}}{x_n!}}|\chi^\pm\otimes\Phi_{n+1}\rangle, \tag{80}$$

$$N|\mathfrak{q};\pm\rangle_1 = \left(\mathcal{N}_{(\star)}(|\mathfrak{q}|)\right)^{-1/2}\sum_{n=0}^{\infty}\frac{\mathfrak{q}^n x_n}{\sqrt{x_n!}}|\chi^{\pm}\otimes\Phi_n\rangle. \tag{81}$$

Furthermore, we have

$$\begin{aligned}
[A,A^\dagger]|\mathfrak{q};\pm\rangle_1 &= \left(\mathcal{N}_{(\star)}(|\mathfrak{q}|)\right)^{-1/2}\sum_{n=0}^{\infty}\mathfrak{q}^n\frac{(x_{n+1}-x_n)}{\sqrt{x_n!}}|\chi^{\pm}\otimes\Phi_n\rangle,\\
[Q,P]|\mathfrak{q};\pm\rangle_1 &= \imath\left(\mathcal{N}_{(\star)}(|\mathfrak{q}|)\right)^{-1/2}\sum_{n=0}^{\infty}\mathfrak{q}^n\frac{(x_{n+1}-x_n)}{\sqrt{x_n!}}|\chi^{\pm}\otimes\Phi_n\rangle\\
&= \imath[A,A^\dagger]|\mathfrak{q};\pm\rangle_1
\end{aligned} \tag{82}$$

where

$$Q = \frac{A+A^\dagger}{\sqrt{2}}, \qquad P = \frac{A-A^\dagger}{\imath\sqrt{2}}. \tag{83}$$

3.3.2 Expectations and uncertainty relation

The expectations of the operators $A, A^\dagger, N, P$ and Q for the QVCS (47) are derived in the quantum state $|\mathfrak{q};\pm\rangle_1$ as follows:

$$\langle A\rangle_{|\mathfrak{q};\pm\rangle_1} = {}_1\langle\mathfrak{q};\pm|A|\mathfrak{q};\pm\rangle_1 = \langle\chi^{\pm}|\mathfrak{q}\chi^{\pm}\rangle = \frac{r}{2}(\cos(\eta)\pm\imath\sin(\eta)\cos(\phi)), \tag{84}$$

$$\langle A^\dagger\rangle_{|\mathfrak{q};\pm\rangle_1} = {}_1\langle\mathfrak{q};\pm|A^\dagger|\mathfrak{q};\pm\rangle_1 = \langle\chi^{\pm}\mathfrak{q}|\chi^{\pm}\rangle = \frac{r}{2}(\cos(\eta)\mp\imath\sin(\eta)\cos(\phi)), \tag{85}$$

$$\langle Q\rangle_{|\mathfrak{q};\pm\rangle_1} = \frac{r}{\sqrt{2}}\cos(\eta), \quad \langle P\rangle_{|\mathfrak{q};\pm\rangle_1} = \pm\frac{r}{\sqrt{2}}\sin(\eta)\cos(\phi), \quad \langle N\rangle_{|\mathfrak{q};\pm\rangle_1} = \frac{1}{2}|\mathfrak{q}|^2 = \frac{r^2}{2}. \tag{86}$$

As other property of the QVCS (47), the following uncertainty relation,

$$(\Delta Q)^2(\Delta P)^2 \geq \frac{1}{4}\left(\left(\mathcal{N}_{(\star)}(|\mathfrak{q}|)\right)^{-1}\sum_{n=0}^{\infty}\frac{(x_{n+1}-x_n)}{x_n!}\langle\mathfrak{q}^n\chi^{\pm}|\mathfrak{q}^n\chi^{\pm}\rangle\right)^2 \tag{87}$$

is satisfied. In the particular situation where $x_n = n, \mathcal{N}_{(\star)}(|\mathfrak{q}|) = 2e^{r^2}$, we readily obtain

$$(\Delta Q)^2(\Delta P)^2 \geq \frac{1}{4}\left(\frac{e^{-r^2}}{2}\sum_{n=0}^{\infty}\frac{r^{2n}}{n!}\right)^2 = \frac{1}{16}. \tag{88}$$

Consider the following annihilation and creation operators $\mathcal{A}, \mathcal{A}^\dagger$ given on $\mathbb{C}^2\otimes\mathfrak{H}$ such that

$$\mathcal{A}|\chi^{\pm}\otimes\Phi_n\rangle = \sqrt{n\omega_{\pm}(\gamma)}|\chi^{\pm}\otimes\Phi_{n-1}\rangle, \quad \mathcal{A}^\dagger|\chi^{\pm}\otimes\Phi_n\rangle = \sqrt{(n+1)\omega_{\pm}(\gamma)}|\chi^{\pm}\otimes\Phi_{n+1}\rangle \tag{89}$$

and set

$$\mathcal{F}_+(\tilde{r}) = \frac{e^{\tilde{r}^2/\omega_+(\gamma)}}{e^{\tilde{r}^2/\omega_+(\gamma)} + e^{\tilde{r}^2/\omega_-(\gamma)}}, \qquad \mathcal{F}_-(\tilde{r}) = \frac{e^{\tilde{r}^2/\omega_-(\gamma)}}{e^{\tilde{r}^2/\omega_+(\gamma)} + e^{\tilde{r}^2/\omega_-(\gamma)}}. \tag{90}$$

In the case of the QVCS (69), we get in the quantum states $|\mathfrak{Q};\pm\rangle_1$ the following relations

$$\begin{aligned} \langle\mathcal{A}\rangle_{|\mathfrak{Q};\pm\rangle_1} &= \tilde{r}\mathcal{F}_\pm(\tilde{r})(\cos(\vartheta) \pm \imath\sin(\vartheta)\cos(\varphi)), \\ \langle\mathcal{A}^\dagger\rangle_{|\mathfrak{Q};\pm\rangle_1} &= \tilde{r}\mathcal{F}_\pm(\cos(\vartheta) \mp \imath\sin(\vartheta)\cos(\varphi)), \end{aligned} \tag{91}$$

$$\langle\mathcal{Q}\rangle_{|\mathfrak{Q};\pm\rangle_1} = \tilde{r}\sqrt{2}\mathcal{F}_\pm(\tilde{r})\cos(\vartheta), \quad \langle\mathcal{P}\rangle_{|\mathfrak{Q};\pm\rangle_1} = \pm\tilde{r}\sqrt{2}\mathcal{F}_\pm(\tilde{r})\sin(\vartheta)\cos(\varphi), \tag{92}$$

$$\langle\mathcal{A}\mathcal{A}^\dagger\rangle_{|\mathfrak{Q};\pm\rangle_1} = (\tilde{r}^2 + \omega_\pm(\gamma))\mathcal{F}_\pm(\tilde{r}), \quad \langle\mathcal{N}\rangle_{|\mathfrak{Q};\pm\rangle_1} = \tilde{r}^2\mathcal{F}_\pm(\tilde{r}), \quad \mathcal{N} = \mathcal{A}^\dagger\mathcal{A}, \tag{93}$$

where $\mathcal{Q} = \frac{1}{\sqrt{2}}(\mathcal{A} + \mathcal{A}^\dagger)$ and $\mathcal{P} = \frac{1}{\imath\sqrt{2}}(\mathcal{A} - \mathcal{A}^\dagger)$.
The expectations of the operators $\mathcal{Q}^2$ and $\mathcal{P}^2$ are obtained as

$$\langle\mathcal{Q}^2\rangle_{|\mathfrak{Q};\pm\rangle_1} = \frac{1}{2}[4\tilde{r}^2\cos^2(\vartheta) + \omega_\pm(\gamma)]\mathcal{F}_\pm(\tilde{r}), \quad \langle\mathcal{P}^2\rangle_{|\mathfrak{Q};\pm\rangle_1} = \frac{1}{2}[4\tilde{r}^2\sin^2(\vartheta) + \omega_\pm(\gamma)]\mathcal{F}_\pm(\tilde{r}). \tag{94}$$

There follow the dispersions given by

$$\begin{aligned} (\Delta\mathcal{Q})^2_{|\mathfrak{Q};\pm\rangle_1} &= 2\tilde{r}^2\mathcal{F}_+(\tilde{r})\mathcal{F}_-(\tilde{r})\cos^2(\vartheta) + \omega_\pm(\gamma)\frac{\mathcal{F}_\pm(\tilde{r})}{2} \\ (\Delta\mathcal{P})^2_{|\mathfrak{Q};\pm\rangle_1} &= 2\tilde{r}^2\mathcal{F}_\pm(\tilde{r})\sin^2(\vartheta)\left[1 - \mathcal{F}_\pm(\tilde{r})\cos^2(\varphi)\right] + \omega_\pm(\gamma)\frac{\mathcal{F}_\pm(\tilde{r})}{2}. \end{aligned} \tag{95}$$

If we take $\omega_+(\gamma) = \omega_-(\gamma)$, then $\mathcal{F}_+(\tilde{r}) = \frac{1}{2} = \mathcal{F}_-(\tilde{r})$. Then,

$$\begin{aligned} (\Delta\mathcal{Q})^2_{|\mathfrak{Q};\pm\rangle_1} &= \frac{1}{4}[\omega_\pm(\gamma) + 2\tilde{r}^2\cos^2(\vartheta)] \\ (\Delta\mathcal{P})^2_{|\mathfrak{Q};\pm\rangle_1} &= \frac{1}{4}\left[\omega_\pm(\gamma) + 4\tilde{r}^2\sin^2(\vartheta)\left(1 - \frac{1}{2}\cos^2(\varphi)\right)\right]. \end{aligned} \tag{96}$$

Assuming that $\tilde{r}^2\cos^2(\vartheta) \geq \frac{1}{2}$ and $\tilde{r}^2\sin^2(\vartheta) \geq \frac{1}{2}$, we get the following relations

$$\begin{aligned} (\Delta\mathcal{Q})^2_{|\mathfrak{Q};\pm\rangle_1} &\geq \frac{1}{4}[\omega_\pm(\gamma) + 1] \geq \frac{1}{4} \\ (\Delta\mathcal{P})^2_{|\mathfrak{Q};\pm\rangle_1} &\geq \frac{1}{4}[\omega_\pm(\gamma) + 1] \geq \frac{1}{4} \end{aligned} \tag{97}$$

which yield $(\Delta\mathcal{Q})^2_{|\mathfrak{Q};\pm\rangle_1}(\Delta\mathcal{P})^2_{|\mathfrak{Q};\pm\rangle_1} \geq \frac{1}{16}$.

3.3.3 Time evolution

The QVCS given in (47) and (69) can be equipped with a parameter τ such that they are rewritten as

$$\begin{aligned} |\mathfrak{q}; \pm, \tau\rangle_1 &= \mathbb{U}(\tau)|\mathfrak{q}; \pm\rangle_1 \\ &= \big(\mathcal{N}_{(\star)}(|\mathfrak{q}|)\big)^{-1/2} \sum_{n=0}^{\infty} \frac{\mathfrak{q}^n}{\sqrt{\rho_{(\star)}(n)}} e^{-i\tau\mathcal{E}_n^{\pm}} |\chi^{\pm} \otimes \Phi_n\rangle, \end{aligned} \tag{98}$$

and

$$|\mathfrak{Q}; \pm, \tau\rangle_1 = \mathbb{U}(\tau)|\mathfrak{Q}; \pm\rangle_1 = (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} (R(n))^{-1/2} \mathfrak{Q}^n e^{-i\tau\mathcal{E}_n^{\pm}} |\chi^{\pm} \otimes \Phi_n\rangle \tag{99}$$

where $\mathbb{U}(\tau) = e^{-i\tau\mathbb{H}_{1SO}^{red}}$. The parameter τ is introduced such that the states (98) and (99), relatively to the time evolution operator $\mathbb{U}(t) = e^{-it\mathbb{H}_{1SO}^{red}}$, fulfill the following property

$$\mathbb{U}(t)|\mathfrak{q}; \pm, \tau\rangle_1 = e^{-it\mathbb{H}_{1SO}^{red}} |\mathfrak{q}; \pm, \tau\rangle_1 = |\mathfrak{q}; \pm, \tau + t\rangle_1, \tag{100}$$

$$\mathbb{U}(t)|\mathfrak{Q}; \pm, \tau\rangle_1 = e^{-it\mathbb{H}_{1SO}^{red}} |\mathfrak{Q}; \pm, \tau\rangle_1 = |\mathfrak{Q}; \pm, \tau + t\rangle_1. \tag{101}$$

Conclusion

We have first investigated, in section 2, a Hamiltonian describing an electron gas in a constant electromagnetic field with Zeeman coupling interaction. A more general Hamiltonian with Rashba and Dresselhaus spin-orbit interactions has been also analysed with a generalization of these both interactions, denoted by V_{SO}, provided. Then, in section 3, we have provided the canonical QVCS related to the studied Hamiltonian and then discussed the VCS construction with their connection with the QVCS. Some interesting physical features of the constructed vectors, arising from the introduction of generalized annihilation and creation operators which allow the definitions on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$ of quadrature operators have been displayed; expectations of these operators and uncertainty relation have been computed. With the method of construction developed here, it is possible to extend the study in the situation of magnetoresistivity and inelastic light-scattering experiments [12] where the interplay between Rashba, Dresselhaus, and Zeeman interactions in a quantum well submitted to an external magnetic field has been studied.

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References

[1] Ulloa S E and Zarea M 2005 *Phys. Rev.* B **72** 085342

[2] Rashba E I 1960 *Sov. Phys. Solid state* **2** 1109

[3] Miller J B, Zumbühl D M, Marcus C M, Lyanda-Geller Y B, Goldhaber-Gordon D, Campman K and Gossard A C 2003 *Phys. Rev. Lett.* **90** 076807

[4] Nitta J, Akazaki T, Takayanagi H and Enoki T 1997 *Phys. Rev. Lett.* **78** 1335

[5] Dresselhaus G 1955 *Phys. Rev.* B **100** 580

[6] Shen S-Q, Bao Y-J, Ma M, Xie X C and Zhang F C 2005 *Phys. Rev.* B **71** 155316

[7] Ben Geloun J and Hounkonnou M N 2007 *J. Math. Phys.* **48** 093505

[8] Ben Geloun J, Govaerts J and Hounkonnou M N 2007 *J. Math. Phys.* **48** 032107

[9] Winkler R 2003 *Spin-orbit Coupling Effects in Two-Dimensional Electron and Hole Systems* (Berlin: Springer-Verlag)

[10] Ali S T and Thirulogasanthar K 2003 *J. Math. Phys.* **44** 5070

[11] Ali S T, Antoine J P and Gazeau J P 2000 *Coherent States, Wavelets and their Generalizations* (New-York: Springer-Verlag)

[12] Lipparini E, Barranco M, Malet Francese and Pi M 2006 *Phys. Rev.* B **74** 115303